A10.1 Gradients of graphs

Before you start

You should be able to:

- find the gradient of the line joining two points.

Why do this?

Aircraft engineers need to know about accelerations so that aircraft can be designed properly.

Objectives

- You can find an estimate for the gradient of a curve at any point by drawing a tangent to the curve.
- You can interpret the gradient of a curve as the rate of change of a quantity.

Get Ready

1 What is the gradient of the line segments which join these points?

a A (2, 3) and B (4, 12)

b C (4, 10) and D (6, 6)

c E (−3, 3) and F (−1, 6)?

2 What does the phrase 'a tangent to a circle' mean?

Key Points

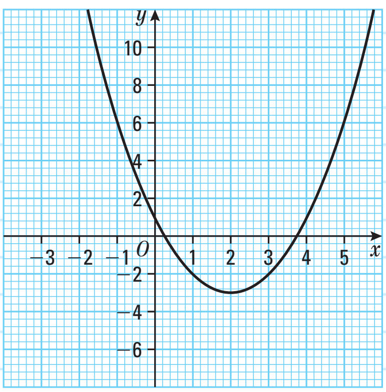
- The tangent at a point P on a graph is the straight line which just touches the graph at the point P.
- The gradient at a point on a graph is the gradient of the tangent to the graph at that point.
- The gradient of the tangent can be found in the same way as the gradient of any straight line.
- For a distance-time graph, the gradient is equal to the speed.
- For a speed-time graph or velocity-time graph, the gradient is equal to the acceleration.
- The acceleration of an object is equal to its rate of change of velocity.

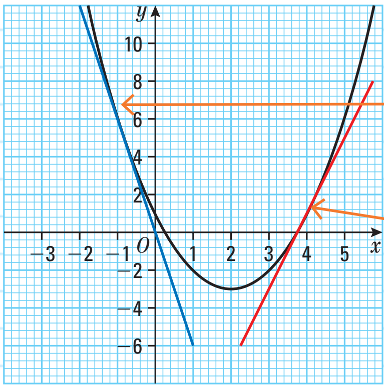
Example 1

Here is the graph of $y = x^2 - 4x + 1$

Work out an estimate for the gradient of the curve at:

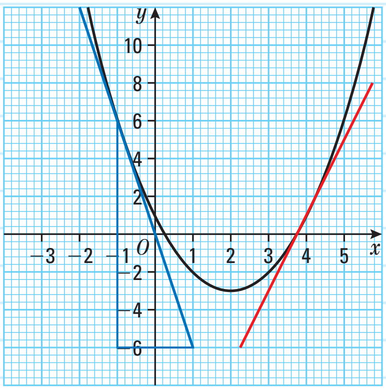
a $x = -1$ **b** $x = 4$





Draw the tangent to the curve at $x = -1$ (blue line)

Draw the tangent to the curve at $x = 4$ (red line)



Draw a triangle with the tangent as hypotenuse and work out $\frac{\text{Difference in } y \text{ values}}{\text{Difference in } x \text{ values}}$

− sign, because the tangent is downwards at the point where $x = -1$

From the diagram, at $x = -1$, gradient of the tangent $= \frac{6 - -6}{-1 - 1} = -6$,

at $x = 4$, gradient of the tangent $= \frac{5 - -3}{5 - 3} = 4$

Note that these are estimates as they depend on drawn tangents

Example 2

Rates of change

The average rate of change of a quantity is: $\frac{\text{Change in quantity}}{\text{time}}$

The water level in a tank was 40 cm at 01:00 and 90 cm at 05:00.

a Work out the average rate of change in the water level.

$\frac{90 - 40}{5 - 1} = 12.5$ cm per hour

$\frac{\text{Change in level}}{\text{time}}$

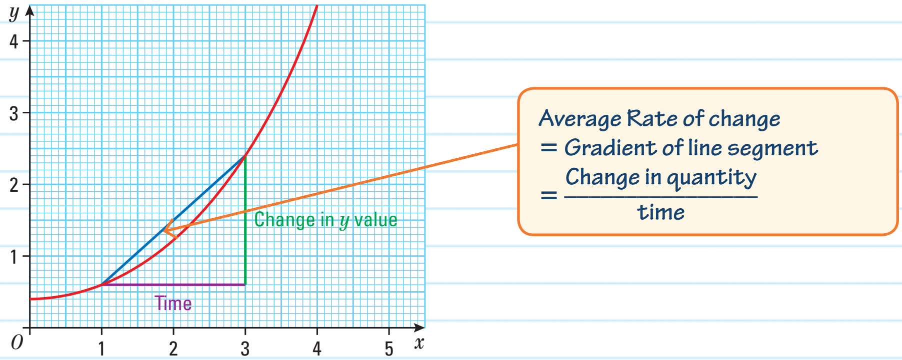
Between 05:00 and 10:00 the water level fell from 90 cm to 30 cm.

b Work out the average rate of change in the water level.

$\frac{\text{Change in level}}{\text{time}} = \frac{30 - 90}{10 - 5} = -12$ cm per hour

− sign because the water level decreases through the time interval

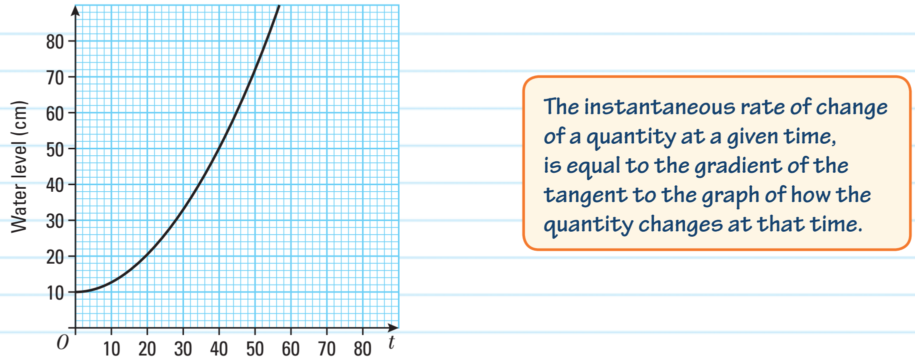
For a graph of a quantity which varies with time, the average rate of change of the quantity is equal to the gradient of the line segment joining the two points over which the quantity changes.



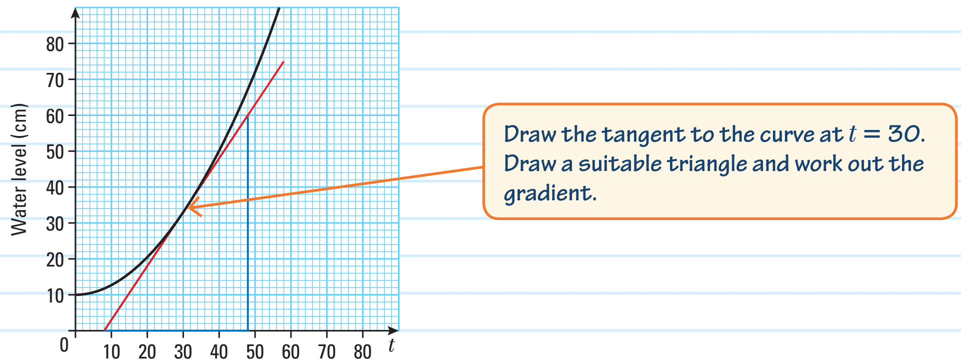
$\frac{2.4 - 0.6}{3 - 1} = 0.9$ units per time

The instantaneous rate of change of a quantity at a given time is equal to the gradient of the tangent to the graph of how the quantity changes at that time.

Example 3 Here is the graph of how the water level changes in a water tank.



Find an estimate for the rate of change of water level at $t = 30$.



Rate of change of water level (at $t = 30$) $= \frac{60 - 0}{48 - 8} = 1.5$ cm per sec

For distance-time graphs, the gradient of the graph at any point gives the speed at that instant.

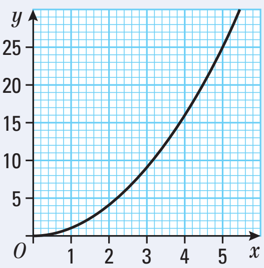
For velocity-time graphs, the gradient of the graph at any point gives the acceleration at that instant.

Exercise 10A

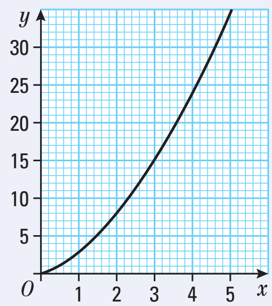
B

1 Find the gradients of the following curves at the values of x given.

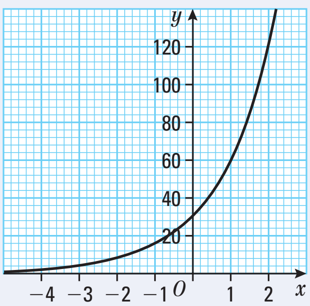
a At $x = 3$



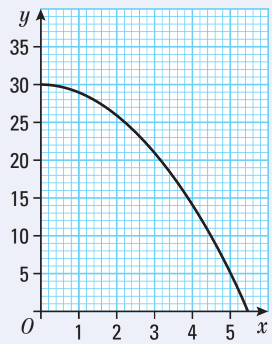
b At $x = 2$



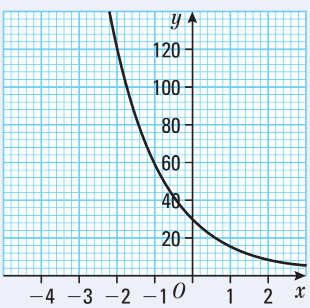
c At $x = -1$



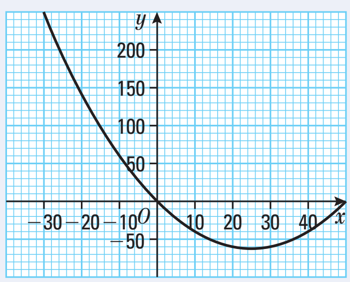
d At $x = 2$



e At $x = -1$



f At **i** $x = -20$ and **ii** $x = 30$



2 Draw the curve with equation $y = x^2$ for values of x from -3 to 3.
Calculate an estimate of the gradient of the curve at these points:

a $(-2,4)$ **b** $(1,1)$

A

3 **a** Copy and complete the table of values for the curve with equation $y = x^2 + 4x$ for values of x from 0 to 5.

x	0	1	2	3	4	5
y		5				45

b Draw the graph.

c Calculate an estimate for the gradient of the graph at $x = 2$.

A

4 a Copy and complete the table of values for the curve with equation $y = 8x - x^2$.

x	0	1	2	3	4	5	6
y		7				15	

b Draw the graph.

c Calculate an estimate for the gradient of the graph:

i at $x = 2$ ii at $x = 4$

5 a Copy and complete the table of values for the curve with equation $y = x^2 - 4x - 5$.

x	−3	−2	−1	0	1	2	3
y		7				−9	

b Draw the graph.

c i Calculate an estimate for the gradient of the graph at $x = 1$.

ii Calculate an estimate for the gradient of the graph at $x = -2$.

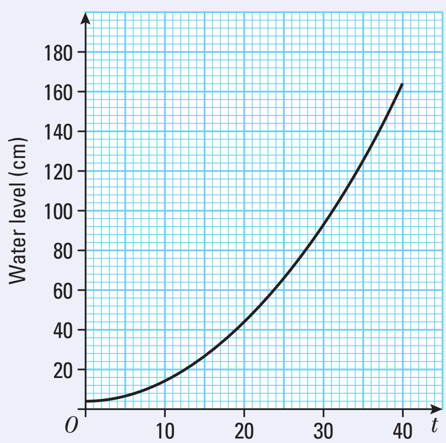
6 a Copy and complete the table of values for the curve with equation $y = \frac{24}{x}$

x	1	2	3	4	5	6
y	24				4.8	

b Draw the graph.

c Calculate an estimate for the gradient of the graph at $x = 4$.

7 The graph shows the water level in a tank.

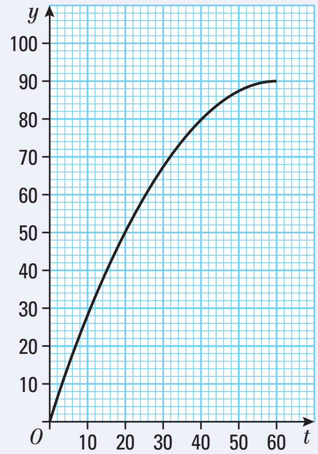


a Work out an estimate for the rate at which the water is rising when $t = 15$

b Work out the average rate of rise of the water level between $t = 10$ and $t = 30$

A

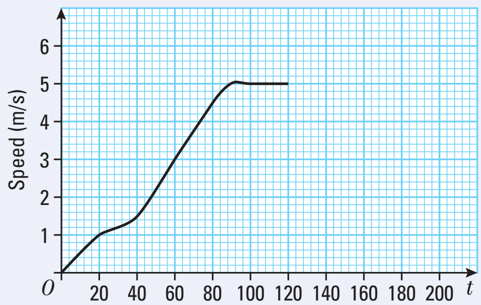
8 The graph shows the distance, y m, that a car has travelled during t seconds.



a Calculate an estimate of the speed of the car at $t = 20$.

b Calculate the average speed of the car between $t = 10$ and $t = 50$.

9 The graph shows the velocity of a train for the first two minutes after it had left a station.



Calculate an estimate of the acceleration of the train after:

a 20 seconds **b** 80 seconds.

For the 3rd minute the train reduces speed at a constant rate until it comes to rest.

Draw the velocity-time graph for the first 3 minutes of the train's journey and find the deceleration of the train.

10 The graph shows the distance that a car has travelled in its first minute measured from a point X on a road.

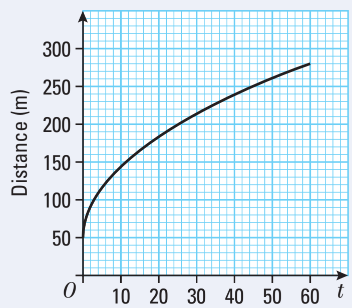
a Calculate an estimate of the speed of the car at $t = 25$.

A van travelling in the same direction along the same road has the distance d metres, it travels in t seconds from the point X, given by the equation $d = 5t$.

b How far ahead of the van was the car initially?

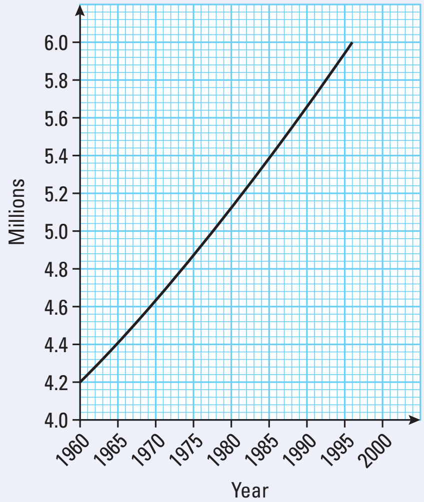
c Describe fully the motion of the van.

d Use the graph to find an estimate of the value of t when the van catches up with the car.



A

11 The graph shows how the population of a city changed with time.



Calculate an estimate of the rate of change of the population at the start of 1980.

Review

- The gradient of a curve at a point is the same as the gradient of the tangent to the curve at that point.
- The gradient of a distance-time graph at a time t is equal to the velocity at time t.
- The gradient of a velocity-time graph at a time t is equal to the acceleration at time t.

Answers

Chapter 10

A10.1 Get Ready answers

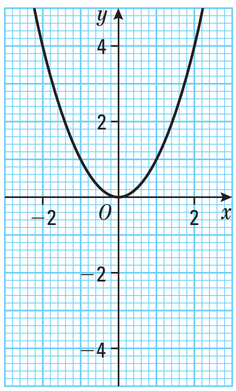
1 **a** 4.5 **b** −2 **c** 1.5

2 A tangent to a circle is a straight line which touches the circle. (More technically, it intersects the circle at two coincident points)

Exercise 10A

1 **a** 6 **b** 6 **c** 10
d −4 **e** −10 **f** **i** −9, **ii** 1

2

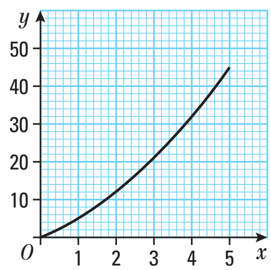


a −4 **b** 2

3 **a**

x	0	1	2	3	4	5
y	0	5	12	21	32	45

b

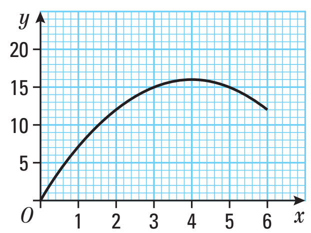


c 8

4 **a**

x	0	1	2	3	4	5	6
y	0	7	12	15	16	15	12

b

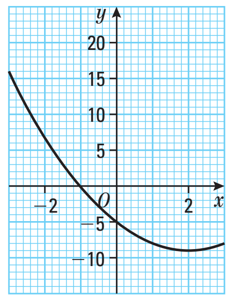


c **i** 4 **ii** 0

5 **a**

x	−3	−2	−1	0	1	2	3
y	16	7	0	−5	−8	−9	−8

b

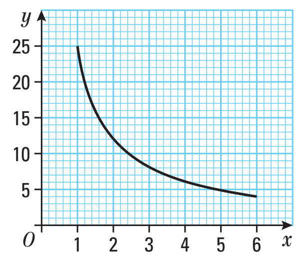


c **i** −2 **ii** −8

6 **a**

x	1	2	3	4	5	6
y	24	12	8	6	4.8	4

b

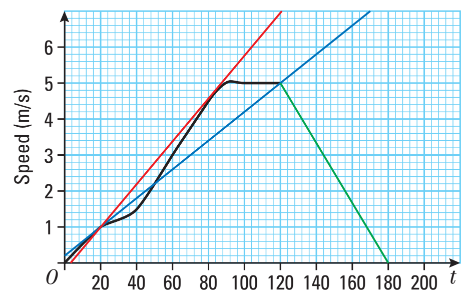


c −1.5

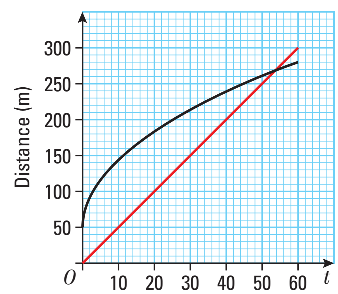
7 **a** 3 cm/sec **b** 4 cm/sec

8 **a** 2 m/s **b** 1.5 m/s

9 **a** 0.04 m/s/s **b** 0.06 m/s/s
c Deceleration is 5 ÷ 60 = 0.083 m/s/s



10 **a** 3 m/s **b** 50 m
c Moving at a constant speed of 5 m/s.
d After 54 seconds



11 0.6 ÷ 10 = 0.06 million per year.